Momentum maps and Noether theorem for generalized Nambu mechanics

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Abstract. In Ref. [2] we proposed a geometric formulation of generalized Nambu mechanics. In the present paper we extend the class of Nambu systems by replacing the stringent condition of constancy of 3-form by closedness. We also explore the connection between continuous groups of symmetries and conservation laws for such systems. The Noether theorem for generalized Nambu systems is formulated by generalizing the notion of momentum map. In this case, a natural choice of dynamical variables for discussion of symmetries is 2-form fields. Thus the generators and the conserved quantities in Noether theorem are best expressed in terms of 2-forms. The connection between the generators and the conserved quantities is illustrated with the example of an axially symmetric top formulated as three dimensional Nambu system.

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1. Introduction

Nambu introduced a generalization of Hamiltonian mechanics in 1973 [1]. In this generalization, points of the phase-space were labeled by a canonical triplet (x, y, z) and the evolution equations were expressed in terms of a pair of Nambu functions $\mathcal{H}_1$ and $\mathcal{H}_2$, through

$$\frac{d\vec{r}}{dt} = \nabla\mathcal{H}_1 \times \nabla\mathcal{H}_2$$

In the subsequent years, the activities related to Nambu systems have steadily developed [3, 4, 5, 6, 7, 8]. In [2] we proposed a geometric formulation of generalized Nambu systems. The motivation behind that work was to provide a framework suitable from dynamical view point. Other generalizations, such as [6], are quite elegant. However, in them, the dynamics on n dimensional system, is governed by $(n-1)$ Nambu functions. i.e., too many integrals of motion are assumed. The framework proposed in [2] involves a $3n$ dimensional manifold M^{3n}, together with a constant (i.e. corresponding to a constant section in the bundle of 3-forms), strictly non-degenerate 3-form $\omega^{(3)}$. The time evolution is governed by a pair (and not $(3n-1)$) of Nambu functions $\mathcal{H}_1, \mathcal{H}_2$. An interesting feature of this formalism is the following. In addition to the three-bracket of functions, a two bracket of 2-forms is defined. This bracket is called Nambu bracket. The Nambu bracket of 2-forms gives rise to a *Lie*

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algebra, whereas, the three bracket of functions does not give rise to an algebra. It was found that the formulation involving 2-forms provides a natural approach to a Nambu system. In the present paper we show that the stringent condition of constancy of 3-form $\omega^{(3)}$ can be moderated by a more general condition of closedness. Also we argue that the formalism involving 2-forms, considered as dynamical variables, is the most suitable to express geometric facts about Nambu systems. Equivalence classes of pairs of functions, which are associated with 2-forms, can be chosen simply by fixing the gauge. The phase point is determined (up to gauge) in terms of 2-forms considered as dynamical variables. We prove Nambu-Darboux theorem in this new setup. we also discuss systematically the connection between symmetries of Nambu systems and conserved 2-forms. In particular, we introduce the notion of "momentum map" and carry out the generalization of symplectic Noether theorem for Nambu systems. Unlike in the case of Hamiltonian systems, here the roles of generator and of conserved quantity are played by 2-forms.

In section 2 we list the main features of geometric formulation developed in [2], the new statement of Theorem 2.1 incorporates more general systems as Nambu systems. In section 3 we introduce momentum maps, symmetries and Noether theorem for Nambu systems and illustrate them with the help of an example of a symmetric top.

2. Nambu Systems

In this section we will briefly review the relevant definitions and results regarding Nambu systems. The Nambu structure was introduced through the following definitions [2]. We also prove that even with *closed* 3-form (instead of *constant*) a Nambu-Darboux coordinates can be obtained.

Definition 2.1. (Nondegenerate form) : Let E be a finite dimensional vector space and let $\omega^{(3)}$ be a 3-form on E i.e.,

$$\omega^{(3)} : E \times E \times E \to \mathbb{R}$$

the form $\omega^{(3)}$ is called a *nondegenerate form* if

$$\begin{aligned}&\forall \;\; non \;\; zero \;\; e_1 \in E \;\; \exists \;\; e_2, e_3 \in E \\ &such \;\; that \;\; \omega^{(3)}(e_1, e_2, e_3) \neq 0\end{aligned}$$

Definition 2.2. (Nambu complement) : Let E be an m dimensional vector space with $m \geq 3$. Let $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E. Let us choose $e_1, e_2, e_3 \in E$ such that $\omega^{(3)}(e_1, e_2, e_3) \neq 0$. Let $P_1 = Span(e_1, e_2, e_3)$, then the *Nambu complement* of P_1 is defined as

$$P_1^{\perp_E} = \{z \in E \;\mid\; \omega^{(3)}(z, z_1, z_2) = 0 \;\; \forall \;\; z_1, z_2 \in P_1\}$$

Definition 2.3. (Strictly nondegenerate form) : Let E be an m dimensional vector space and $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E, the $\omega^{(3)}$ is called *strictly non-degenerate* if for each non zero $e_1 \in E$ $\exists$ a two dimensional subspace $E_1 \subset E$ such that

(i) $\omega^{(3)}(e_1, x_1, x_2) \neq 0$ $\forall$ linearly independent $\{e_1, x_1, x_2\}$ where $x_1, x_2 \in F_1$ and $F_1 = Span(e_1 + E_1)$.

(ii) $\omega^{(3)}(e_1, z_1, z_2) = 0 \;\forall\; z_1, z_2 \in F_1^{\perp_E}$.

Using the definition of Strictly nondegenerate 3-form, we define the Nambu manifold as

Definition 2.4. (Nambu Manifold) : Let M^{3n} be a $3n$-dimensional C^∞ manifold and let $\omega^{(3)}$ be a 3-form field on M^{3n} such that $\omega^{(3)}$ is completely anti-symmetric, closed and strictly nondegenerate at every point of M^{3n} then the pair $(M^{3n}, \omega^{(3)})$ is called a *Nambu manifold*.

Remark: Note the change in the definition from that of [2]. The constant 3-form is now replaced by closed 3-form. An analog of canonical transformation was defined in [2].

Definition 2.5. (Canonical transformation) : Let $(M^{3n}, \omega^{(3)})$ and $(N^{3n}, \rho^{(3)})$ be Nambu manifolds. A C^∞ mapping $F : M^{3n} \to N^{3n}$ is called *canonical transformation* if $F^*\rho^{(3)} = \omega^{(3)}$, where F^* is the pullback of F.

Theorem 2.1. (Nambu-Darboux theorem) : Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold then at every point $p \in M^{3n}$, there is a chart (U, ϕ) in which $\omega^{(3)}$ is written as

$$\omega^{(3)}|_U = \sum_{i=0}^{n-1} dx_{3i+1} \wedge dx_{3i+2} \wedge dx_{3i+3}$$

where $(x_1, x_2, x_3, \ldots, x_{3(n-1)+1}, x_{3(n-1)+2}, x_{3(n-1)+3})$ are local coordinates on U described by ϕ.

Proof: We wish to prove that there exists a canonical transformation which maps any coordinate system to a system in which $\omega^{(3)}$ has the required form. Here we introduce the following notation. Without loss of generality we prove the theorem on a vector space E with point p as the *null vector* of the vector space E. Let $\omega_1 = \omega^{(3)}(0)$ be a constant 3-form on this chart. We define a difference form $\tilde{\omega} = \omega_1 - \omega^{(3)}$. Let

$$\begin{aligned} \omega_t &= \omega^{(3)} + t\tilde{\omega} \quad 0 \le t \le 1 \\ &= (1-t)\omega^{(3)} + t\omega_1 \end{aligned}$$

At $x = 0$ we have $\tilde{\omega} = 0$. Hence, $\omega_t(0) = \omega^{(3)}(0)$ is strictly non-degenerate $\forall\; 0 \le t \le 1$. Also for all x the form ω_t is strictly non-degenerate at $t = 0$ and $t = 1$.

To show that ω_t is strictly-non-degenerate for $0 < t < 1$.
Consider a three dimensional subspace P of E. There are two cases
Case I: If $\omega^{(3)}|_P = 0 \Rightarrow \omega_1|_P = 0 \Rightarrow \omega_t|_P = 0$
Case II: $\omega^{(3)}|_P \neq 0 \Rightarrow \omega_1|_P \neq 0 \Rightarrow \exists$ some neighborhood around $p = 0$ such that $\omega_t|_P \neq 0$ since ω_t is a continuous field.

Since $\omega^{(3)}$ is strictly non-degenerate there are only two such cases. So ω_t is also strictly non-degenerate.

$\omega^{(3)}$ is closed $\Rightarrow \tilde{\omega}$ is closed.
Hence locally, by the Poincaré lemma, $\tilde{\omega} = d\alpha$ where α is a 2-form so chosen that $\alpha(0) = 0$.

Since ω_t is strictly-non-degenerate we have n three dimensional subspaces of E (say P_i, $i = 1, \ldots, n$) on which $\omega_t \neq 0$. These subspaces are Nambu complements of

each other. These are the only subspaces on which $\omega_t \neq 0$. Hence, there exist vector fields X_{t_i} such that

$$i_{X_{t_i}} \omega_t|_{P_i} = -\alpha|_{P_i} \quad \forall \;\; i = 1, \ldots, n$$

Since $E = P_1 \oplus \cdots \oplus P_n$
$X_t = \sum_{i=0}^{n-1} X_{t_i}$ which is zero at $x = 0$. Let F_t be the integral curve of X_t

$$\begin{aligned}\frac{d}{dt}(F_t^* \omega_t) &= F_t^*(L_{X_t}\omega_t) + F_t^* \frac{d}{dt}\omega_t \\ &= F_t^* d i_{X_t}\omega_t + F_t^* \tilde{\omega} \\ &= F_t^*(-d\alpha + \tilde{\omega}) = 0\end{aligned}$$

Hence, F_1 provides the required coordinate transformation

□

The theorem establishes the existence of a canonical coordinate system, in which $\omega^{(3)}$ has a "Normal form".

Let $\mathcal{T}_k^0(M^{3n})$ denote a bundle of k-forms on M^{3n}, $\Omega_k^0(M^{3n})$ denote the space of k-form fields on M^{3n} and $\mathcal{X}(M^{3n})$ denote the space of vector fields on M^{3n}. Now for a given vector field X on M^{3n}, we define the inner product of X with k-form (or contraction of a k-form by X) as

$$(i_X \eta^{(k)})(a_1, \ldots, a_{k-1}) = \eta^{(k)}(X, a_1, \ldots, a_{k-1})$$

where $\eta^{(k)} \in \Omega_k^0(M^{3n})$ and $a_1, \ldots, a_{k-1} \in \mathcal{X}(M^{3n})$

The operations of Lowering and Raising were defined in [2] as follows. The Lowering map $\flat : \mathcal{X}(M^{3n}) \to \Omega_2^0(M^{3n})$ defined by $X \mapsto X^\flat = i_X \omega^{(3)}$, and the Raising map $\sharp : \Omega_2^0(M^{3n}) \to \mathcal{X}(M^{3n})$, is defined by the following prescription. Let α be a 2-form and α_{ij} be its components in Nambu-Darboux coordinates, then the components of $\alpha^\sharp$ are given by

$$\alpha^{\sharp^{3i+p}} = \frac{1}{2} \sum_{l,m=1}^{3} \varepsilon_{plm} \alpha_{3i+l \quad 3i+m} \tag{1}$$

where $0 \leq i \leq n-1$, $p = 1, 2, 3$ and ε_{plm} is the Levi-Cevità symbol.

Here we have an important remark to make. On the space $\mathcal{T}_{2_x}^0(M^{3n})$, the space of 2-forms at $x \in M^{3n}$ the $\sharp$ defines an equivalence relation such as $\omega_1^{(2)}(x) \sim \omega_2^{(2)}(x)$ if $(\omega_1^{(2)})^\sharp(x) = (\omega_2^{(2)})^\sharp(x)$, where $\omega_1^{(2)}, \omega_2^{(2)} \in \mathcal{T}_{2_x}^0(M^{3n})$.

Equation (1) provides a relation between 2-forms and vector fields. The following theorem, established in [2], provides conditions on 2-form, under which the associated flow preserves the Nambu structure.

Theorem 2.2. Let $\beta^{(2)} \in \Omega_2^0(M^{3n})$, and f^t be a flow corresponding to $\beta^{(2)^\sharp}$, i.e., $f^t : M^{3n} \to M^{3n}$ such that

$$\left.\frac{d}{dt}\right|_{t=0} (f^t x) = (\beta^{(2)^\sharp})x \;\; \forall x \in M^{3n}$$

Then the form $\omega^{(3)}$ is preserved under the action of $\beta^{(2)^\sharp}$ iff $d(\beta^{(2)^\sharp})^\flat = 0$. i.e., ${f^t}^* \omega^{(3)} = \omega^{(3)}$ iff $d(\beta^{(2)^\sharp})^\flat = 0$

From theorem 2.2 it follows that the vector fields preserving the Nambu structure can be obtained from two functions, called Nambu functions, as in [2]. These Nambu functions were shown to be constants of motion. At this stage, the following definitions were introduced.

Definition 2.6. (Nambu vector field) : Let $\mathcal{H}_1$, $\mathcal{H}_2$ be real valued C^∞ functions (Nambu functions) on $(M^{3n}, \omega^{(3)})$ then N is called *Nambu vector field* corresponding to $\mathcal{H}_1, \mathcal{H}_2$ if

$$N = (d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp$$

Definition 2.7. (Nambu system) : A four tuple $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ is called *Nambu system.*

Definition 2.8. (Nambu phase flow) : Let $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ be a Nambu system and let the $g^t : M^{3n} \to M^{3n}$ be one parameter family of group of diffeomorphisms. The flow corresponding to g^t is *Nambu flow* if

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}(g^t\mathbf{x}) &= (d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp \mathbf{x} \quad \forall\, \mathbf{x} \in M^{3n} \\ &= N\mathbf{x} \end{aligned}$$

It can be shown that g^t preserves $\omega^{(3)}$ [2].

Definition 2.9. (Nambu bracket) : Let $\omega_a^{(2)}$ and $\omega_b^{(2)}$ be 2-forms then the *Nambu bracket* is a map $\{,\} : \Omega_2^0(M^{3n}) \times \Omega_2^0(M^{3n}) \to \Omega_2^0(M^{3n})$ given by

$$\{\omega_a^{(2)}, \omega_b^{(2)}\} = [\omega_a^{(2)\sharp}, \omega_b^{(2)\sharp}]^\flat$$

where $[,]$ is Lie bracket of vector fields.

From definition 2.9 it follows that the bracket of 2-forms provides a Lie algebra of 2-forms.

In case of Hamiltonian systems, the Poisson bracket generates the Hamiltonian vector field. A similar result was proved in [2]. This assures that if α and β are 2-forms, if $\alpha^\sharp$ is a Nambu vector field and if $\alpha' = (\alpha^\sharp)^\flat$ and $\beta' = (\beta^\sharp)^\flat$ then

$$\{\alpha, \beta\} = L_{\alpha'^\sharp}\beta'$$

where $L_{\alpha'^\sharp}$ denote the Lie derivative along the vector field $\alpha'^\sharp$.

Definition 2.10. (Nambu bracket for functions) : Consider a Nambu manifold $(M^{3n}, \omega^{(3)})$ and let f, g, h be C^∞ functions on M^{3n} then *Nambu bracket for functions* is given by

$$\{f, g, h\} = L_{(dg\wedge dh)^\sharp} f = i_{(dg\wedge dh)^\sharp} df$$

We wish to emphasize that in case of the Hamiltonian systems, the bracket of functions is equivalent to the bracket of 1-forms. In the present formulation, the bracket of 2-forms and the bracket of functions are not equivalent, but they are related to each other as given in the following proposition which is proved in [2].

Proposition 2.1. Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold and let f, g, h_1, h_2 be C^∞ functions satisfying $(df \wedge dg)^{\sharp^\flat} = df \wedge dg$ and $(dh_1 \wedge dh_2)^{\sharp^\flat} = dh_1 \wedge dh_2$ then

$$\{dh_1 \wedge dh_2, df \wedge dg\} = d\{f, h_1, h_2\} \wedge dg + df \wedge d\{g, h_1, h_2\}$$

3. Nambu momentum maps

3.1. Forms as dynamical variables

Let us recall the Hamiltonian formalism. In that case, the phase space is a symplectic manifold $(M^{2n}, \omega^{(2)})$. i.e., an even dimensional manifold equipped with a closed non-degenerate 2-form. The vector field are in one to one correspondence with the 1-forms as $\alpha = \omega^{(2)}(\cdot, X_\alpha)$ where $\alpha \in \Omega_2^0(M^{2n})$ and $X_\alpha \in \mathcal{X}(M^{2n})$ [9]. Hence, Hamiltonian systems can be completely formulated using 1-forms considered as dynamical variables. Consequently the conservation laws and symmetry are also formulated in terms of 1-form. The relation between the 1-form and functions involve a one parameter family of freedom (e.g.: adding a constant to Hamiltonian does not change the equations of motion). Thus, for convenience, the constants of motion are considered as functions rather than 1-forms.

Now, in the Nambu framework, it is natural to consider 2-form fields as dynamical variables. Because, as mentioned above, there is a correspondence between vector fields and classes of 2-forms [2]. In fact, it is possible to obtain other types of dynamical variables from 2-forms using the following procedure. As shown in [2], the space of equivalence classes of 2-forms at each point is a 3n-dimensional space. A set of $3n$ independent vector fields, and subsequently $3n$ local coordinates can be obtained from $3n$ independent 2-forms. Thus all types of dynamical variables (e.g. functions) can be recovered, in principle, from the convenient set of 2-forms.

We call a pair of functions (f, g) to be equivalent to another pair (h, p) if $df \wedge dg = dh \wedge dp$. So the equivalence classes of pairs of functions are locally associated with closed 2-forms. Any closed 2-form can be represented by giving a representative pair in the equivalence classes of pairs of functions. The choice of representative pair is called “fixing the gauge” (See [2]).

3.2. Nambu momentum map

Before formalizing the concept of Nambu momentum mapping we state the standard definitions.

Definition 3.1. (Invariants [9]) : Let M^{3n} be a manifold and $X \in \mathcal{X}(M^{3n})$. Let α be a k-form on M^{3n}. We call α an invariant k-form with respect to X iff $L_X\alpha = 0$.

Definition 3.2. (Group action on manifold [9]) : Let M be a manifold and let G be a Lie group. A *group action on manifold* M is a smooth mapping $\Phi_G : G \times M \to M$ such that

(i) $\Phi_G(e, x) = x$

(ii) $\Phi_G(g, \Phi_G(h, x)) = \Phi_G(gh, x)\ \forall\ g, h \in G$ and $x \in M$

Definition 3.3. (Infinitesimal generator [9]) : Let $\Phi_G : G \times M \to M$ be action of group G on manifold M. For $\xi \in \boldsymbol{g}$ where $\boldsymbol{g}$ is Lie algebra of G, the map $\Phi_G^\xi : \mathbb{R} \times M \to M$, defined by $\Phi_G^\xi(t, x) = \Phi_G(\exp(t\xi), x)$ is an $(\mathbb{R}, +)$ action on M. The *infinitesimal generator*, $\xi_M \in \mathcal{X}(M)$, of action corresponding to ξ is

$$\xi_M(x) = \frac{d}{dt}\Phi_{\exp(t\xi)}(x)\Big|_{t=0}$$

Before proceeding further with Nambu-Noether theorem we briefly recall the corresponding ideas in Hamiltonian dynamics [9]. A group action on symplectic manifold is called a symplectic action if it preserves the symplectic structure. A $\boldsymbol{g}^*$ valued map, (where $\boldsymbol{g}^*$ is dual of Lie algebra), on symplectic manifold is called a momentum map, provided it satisfies certain consistency conditions as shown in [9].

We now develop similar ideas for Nambu systems.

Definition 3.4. (Nambu action) : Let G be a Lie group and let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold. Let Φ_G be the action of G on M^{3n}. Φ_G is called *Nambu action* if

$$\Phi_G^* \omega^{(3)} = \omega^{(3)}$$

i.e., Φ_G induces a Nambu canonical transformation.

We introduce a 2-form valued quantity called momentum, which is a generator of the group action, associated with each one-parameter group of symmetry. Here we define the "Nambu momentum map":

Definition 3.5. (Nambu momentum maps) : Let G be a Lie group and let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold, let Φ_G be a Nambu action on M^{3n}. Then the mapping $\mathbf{J} \equiv (J_1, J_2) : M^{3n} \to \boldsymbol{g}^* \times \boldsymbol{g}^*$ is called *Nambu momentum maps* provided for every $\xi \in \boldsymbol{g}$

$$(d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi))^\sharp = \xi_{M^{3n}}$$

where $\hat{J}_1(\xi) : M^{3n} \to \mathbb{R}$, $\hat{J}_2(\xi) : M^{3n} \to \mathbb{R}$, defined by $\hat{J}_1(\xi)(x) = J_1(x)\xi$ and $\hat{J}_2(\xi)(x) = J_2(x)\xi \ \forall x \in M^{3n}$ and $\xi_{M^{3n}}$ is infinitesimal generator of the action Φ_G.

Remark:

(i) Traditionally the term momentum map has come from the conjugate momenta. Here we can not associate any such meaning to this quantity. We call it as Nambu momentum map because under symetry the 2-form constructed from Nambu momentum map is preserved under Nambu flow (see Theorem 3.1) like the usual momentum.

Definition 3.6. (Nambu G-space) : The five tuple $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ is called *Nambu G-space.*

The following proposition establishes the consistency between the bracket of 2-forms and the Nambu momentum maps.

Proposition 3.1. Let $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ be a Nambu G-space and $\xi, \eta \in \boldsymbol{g}$ then

$$(d\hat{J}_1([\xi,\eta]) \wedge d\hat{J}_2([\xi,\eta]))^\sharp = \{d\hat{J}_1(\eta) \wedge d\hat{J}_2(\eta), d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi)\}^\sharp$$

i.e., The following diagram commutes

$$\begin{tikzcd} \Omega^0_2(M^{3n}) \arrow[r, "\sharp"] & \mathcal{X}(M^{3n}) \\ \boldsymbol{g} \arrow[u, "{\hat{J}_1, \hat{J}_2}"] \arrow[ur, "\xi \mapsto \xi_{M^{3n}}"'] & \end{tikzcd}$$

Proof:

$$\begin{aligned}\{d\hat{J}_1(\eta)\wedge d\hat{J}_2(\eta), \quad & d\hat{J}_1(\xi)\wedge d\hat{J}_2(\xi)\}^\sharp \\ &= [(d\hat{J}_1(\eta)\wedge d\hat{J}_2(\eta))^\sharp, (d\hat{J}_1(\xi)\wedge d\hat{J}_2(\xi))^\sharp] \\ &= [\eta_{M^{3n}}, \xi_{M^{3n}}] \\ &= -[\eta,\xi]_{M^{3n}} \\ &= (d\hat{J}_1([\xi,\eta])\wedge d\hat{J}_2([\xi,\eta]))^\sharp\end{aligned}$$

□

Remark:

(i) Infact, if one starts with bracket of 2-form as more basic quantity than the closed, strictly non-degenerate 3-form $\omega^{(3)}$, then the Proposition 3.1 can be used as the definition of the Nambu momentum maps. The equivalence of such framework with that of current one is not yet established.

In the following examples, we explicitly construct Nambu momentum maps for the group actions of $SO(3)$ and $SP(2)$. They also illustrate the consistency condition stated in Proposition 3.1

Example: Consider action of $SO(3)$ on the Nambu manifold $(\mathbb{R}^3, dx\wedge dy\wedge dz)$, where (x,y,z) are the Nambu-Darboux coordinates on $\mathbb{R}^3$. The Lie algebra $\boldsymbol{g} = \mathbb{R}^3$. Let e_1, e_2, e_3 be a basis of $\boldsymbol{g}$ satisfying the bracket conditions.

$$[e_1,e_2]=e_3, \;\; [e_2,e_3]=e_1, \;\; [e_3,e_1]=e_2$$

Let f_1, f_2, f_3 be a dual basis of $\boldsymbol{g}^*$ corresponding to e_1, e_2, e_3.

Let $\vec{r}\equiv xf_1+yf_2+zf_3, \vec{\rho}\equiv \frac{1}{2}(y^2+z^2)f_1+\frac{1}{2}(x^2+z^2)f_2+\frac{1}{2}(y^2+x^2)f_3 \in \boldsymbol{g}^*$ then the Nambu momentum maps in the Nambu-Darboux coordinates are $J_1(x,y,z)=\vec{r}$ and $J_2(x,y,z)=\vec{\rho}$. The 2-forms obtained from these are

$$\begin{aligned}d\hat{J}_1(e_1)\wedge d\hat{J}_2(e_1) &\equiv L_1 = ydx\wedge dy + zdx\wedge dz \\ d\hat{J}_1(e_2)\wedge d\hat{J}_2(e_2) &\equiv L_2 = -xdx\wedge dy + zdy\wedge dz \\ d\hat{J}_1(e_3)\wedge d\hat{J}_2(e_3) &\equiv L_3 = -xdx\wedge dz - ydy\wedge dz\end{aligned}$$

Little algebra verifies that $\{L_1,L_2\}=-L_3$, $\{L_2,L_3\}=-L_1$ and $\{L_3,L_1\}=-L_2$. This establishes the consistency with Proposition 3.1.

Example: Consider the action of $SP(2,\mathbb{R})$ on $(\mathbb{R}^3, dx\wedge dy\wedge dz)$ defined by $x\mapsto x$ and $(y,z)\mapsto A\cdot(y,z)$ where $A\in SP(2,\mathbb{R})$ and (x,y,z) are the Nambu-Darboux coordinates on $\mathbb{R}^3$. It is easy to check that the action of $SP(2,\mathbb{R})$ is a Nambu action†. The algebra $sp(2,\mathbb{R})$ is three dimensional. Let

$$e_1=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, e_2=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, e_3=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

be a basis of $sp(2,\mathbb{R})$. The bracket relations are

$$[e_1,e_2]=-2e_3, [e_2,e_3]=-2e_1, [e_3,e_1]=2e_2$$

† It follows from the fact that the $SP(2,\mathbb{R})$ preserves area on the two dimensional plane (y,z) and the action is identity on the orthogonal x direction. Hence, the three volume is preserved under this action.

Let f_1, f_2, f_3 be a dual basis of $\boldsymbol{g}^*$ corresponding to e_1, e_2, e_3.

The Nambu momentum maps in Nambu-Darboux coordinates are

$$J_1(x,y,z) = zf_1 + yf_2 + xf_3 \in sp^*(2,\mathbb{R})$$
$$J_2(x,y,z) = (x^2 - y^2)f_1 + (x^2 + z^2)f_2 + (y^2 - z^2)f_3 \in sp^*(2,\mathbb{R})$$

Let $A_i = d\hat{J}_1(e_i) \wedge d\hat{J}_2(e_i), \;\; i = 1,2,3$. Thus

$$A_1 = -2xdx \wedge dz + 2ydy \wedge dz$$
$$A_2 = -2xdx \wedge dy + 2zdy \wedge dz$$
$$A_3 = 2ydx \wedge dy - 2zdx \wedge dz$$

It is easy to check that A_1, A_2, A_3 satisfy the consistency condition established in proposition 3.1. It so happens that the action of $SP(2)$ just displaces the points of $\mathbb{R}^3$ on some conic section.

In analogy with the notion of symplectic symmetry, we now introduce Nambu Lie symmetry.

Definition 3.7. (Nambu Lie symmetry) : Consider a Nambu system $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$. Let $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ be a Nambu G-space. We call Φ_G a *Nambu Lie symmetry* of the Nambu system if

$$\Phi_G^*(d\mathcal{H}_1 \wedge d\mathcal{H}_2) = (d\mathcal{H}_1 \wedge d\mathcal{H}_2)$$

We now come to the main theme of this paper. We first prove the theorem which establishes a connection between the symmetries and invariants of Nambu systems, i.e., Noether theorem for Nambu systems and then elaborate the idea of the theorem, by giving an example of Symmetric top.

Theorem 3.1. (Nambu-Noether theorem): Consider a Nambu system ($M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2$) where $\mathcal{H}_1, \mathcal{H}_2$ are so chosen that $d\mathcal{H}_1 \wedge d\mathcal{H}_2 = (d\mathcal{H}_1 \wedge d\mathcal{H}_2)^{\sharp\flat}$. Let this system be a Nambu G-space $(M^{3n}, \omega^{(3)}, \Phi_G, J_1, J_2)$ where J_1, J_2 are so chosen that $d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi) = (d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi))^{\sharp\flat} \; \forall \xi \in \boldsymbol{g}$. If Φ_G is Nambu Lie symmetry of this system then

$$L_{(d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp}\Big(d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi)\Big) = 0$$

i.e., $d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi)$ is conserved by the Nambu flow.

Proof: Let $g^t = \exp(t\xi)$ be a one parameter subgroup of G. In view of the choice of $\mathcal{H}_1$, $\mathcal{H}_2$ and Nambu Lie symmetry it follows that

$$\begin{aligned}
0 &= \frac{d}{dt}(\Phi^*_{\exp(t\xi)}(d\mathcal{H}_1 \wedge d\mathcal{H}_2))\Big|_{t=0} \\
&= \Phi_e^*(L_{\xi_{M^{3n}}}(d\mathcal{H}_1 \wedge d\mathcal{H}_2)) \\
&= \Phi_e^*\{\xi^\flat_{M^{3n}}, d\mathcal{H}_1 \wedge d\mathcal{H}_2\}
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= \Phi_e^*\{d\mathcal{H}_1 \wedge d\mathcal{H}_2, \xi^\flat_{M^{3n}}\} \\
&= L_{(d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp}(d\hat{J}_1(\xi) \wedge d\hat{J}_2(\xi))
\end{aligned}$$

which follow from the definition of Nambu momentum maps and the choice of $\hat{J}_1$ and $\hat{J}_2$.

$\square$

Remark:

In Hamiltonian dynamics the vector field are in one to one correspondence with the 1-forms as $\alpha = \omega^{(2)}(\cdot, X_\alpha)$ where $\alpha \in \Omega_2^0(M^{2n})$ and $X_\alpha \in \mathcal{X}(M^{2n})$ [9]. Since 1-forms are locally related to the functions, up to a constant, it is traditional to consider functions as generators and conserved quantities. However, 1-forms play role of generators. In the present framework the roles of generator and of conserved quantity is played by 2-form.

Example: Consider the action of $SO(2)$ on the manifold $(\mathbb{R}^3, dx \wedge dy \wedge dz)$ defined by $x \mapsto x$, $(y, z) \mapsto (y\cos(\phi) + z\sin(\phi), -y\sin(\phi) + z\cos(\phi))$ where ϕ is angle of rotation around x axis and (x, y, z) are the Nambu-Darboux coordinates on $\mathbb{R}^3$. The Lie algebra of $SO(2)$ is $so(2) = \mathbb{R}$. Let e_1 be the basis of $so(2)$. Let f_1 be a dual basis corresponding to e_1. The Nambu momentum maps corresponding to the action of $SO(2)$ in Nambu-Darboux coordinates are

$$J_1(x, y, z) = xf_1$$
$$J_2(x, y, z) = (\frac{1}{2}(y^2 + z^2))f_1$$

The the 2-form

$$d\hat{J}_1(e_1) \wedge d\hat{J}_2(e_1) \equiv L_1 = ydx \wedge dy + zdx \wedge dz$$

Now consider a Nambu system for a symmetric top. Let (x, y, z) be the components of angular momentum of the top in the body frame. The Nambu functions are

$$\mathcal{H}_1 = \frac{1}{2}(\frac{x^2}{I_x^2} + \frac{y^2}{I_y^2} + \frac{z^2}{I_y^2})$$
$$\mathcal{H}_2 = x^2 + y^2 + z^2$$

It follows that $L_{L_1^\sharp}(d\mathcal{H}_1 \wedge d\mathcal{H}_2) = 0$. It is easy to see that $L_{(d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp} L_1 = 0$. The L_1 is conserved.

4. Conclusions

In the present paper we have enlarged the class of Nambu systems. This is achieved by replacing the stringent condition of constancy of 3-form by the closedness of the 3-form. We have also developed the notion of symmetry of a Nambu system. In the geometric formulation of Nambu mechanics, it is natural to consider 2-forms as dynamical variables. Therefore, the notion of momentum maps in this framework is formulated in terms of 2-forms. The connection between continuous groups of symmetries and the conservation laws, i.e., the Noether theorem, is proved in the present work. The result is illustrated with the help of an example of an axially symmetric 3-dimensional Nambu system.

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